B. Math. (Hons.) 2nd year
First Midsemestral Examination
Algebra III
9th September 2019
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Total Marks 60

Q 1. Let R be a finite ring. Prove the following:
(a) There exists an infinite sequence $n_1 < n_2 < \cdots$ such that $a^{n_i} = a^{n_j}$ for all i, j and all $a \in R$.
(b) In addition, if R is also commutative with unity, then every prime ideal is maximal.

OR

Let $n \geq 3$ and let R be the ring of $n \times n$ strictly lower triangular matrices with integer entries.
(a) Find with proof an example of a proper, non-zero left ideal of R.
(b) Find the smallest positive integer N such that $r^N = 0$ for all $r \in R$.

Q 2. Let R be the ring of real-valued continuous functions on $[0, 1]$.
(a) Prove that the map $f \mapsto \int_0^1 f(x)dx$ from R to $\mathbb{R}$ is not a ring homomorphism.
(b) Prove that the set $I = \{f \in R : f(1/4) = 0 = f(1/2)\}$ is an ideal. Is it a prime ideal? Why?

OR

Let R be a ring in which for each nonzero element a, one has a unique corresponding element a^* such that $aa^*a = a$. Prove that R must have a unity, and must be a division ring.

Q 3. In the Gaussian ring $R = \mathbb{Z}[i]$, prove that the ideal $I = (1 + 2i), J = (1 - 2i)$ are comaximal. Further, show that the quotient ring $\mathbb{Z}[i]/5\mathbb{Z}[i]$ is isomorphic to $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

OR

For a given non-empty set X, consider the power set $P(X)$ as a ring under the operations $A + B := (A \setminus B) \cup (B \setminus A)$, $A.B := A \cap B$. Prove that for any proper subset Y of X, the ring $P(Y)$ is an ideal of $P(X)$ and that the quotient ring $P(X)/P(Y)$ is isomorphic to the ring $P(X \setminus Y)$.

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Q 4. Let R be a ring with unity. Suppose e is an idempotent in R that is in the center. Prove that $Re, R(1 - e$ are two-sided ideals of R and $R = Re \oplus R(1 - e)$. Further, show that $Re, R(1 - e)$ are rings with unity. Deduce that any finite Boolean ring is isomorphic to a finite product $R \cong \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$.

OR

Consider the ring $R = \mathbb{Z}_9[[X]]$ of formal power series. Prove:
(a) it has uncountably many non-units;
(ii) it has only finitely many nilpotent elements.

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Q 5. Let $R = \{p/q \in \mathbb{Q} : (30p, q) = 1\}$.
(a) Find all the maximal ideals of R.
(b) Find a multiplicative subset of $\mathbb{Z}$ such that $R = S^{-1}\mathbb{Z}$.

OR

Let R be a commutative ring with unity.
(a) Find the largest multiplicative subset S of R for which the ring homomorphism $r \mapsto [r/1]$ from R to $S^{-1}R$ is injective.
(b) If $S^{-1}R$ denotes the localization of R at a minimal prime ideal P (that is, $P = R - S$ is a prime ideal that does not properly contain any prime ideals of R), find all the prime ideals ideal Q of $S^{-1}R$.

Q 6. Let R be a commutative ring with unity.
(a) If P is a prime ideal containing the intersection of ideals $I_1, \cdots, I_n$, prove that P contains some I_i.
(b) If I is an ideal contained in the union of prime ideals $P_1, \cdots, P_m$, prove that I is contained in some P_j.

OR

Let R be an integral domain. Show that $R = \cap_M R_M$ where R_M is the localization of R at M, the intersection is taken in the quotient field of R and is over all maximal ideals of R.
Hint. In the quotient field of R, write any element of the intersection as $x = a_M/b_M$ with $b_M \notin M$ for each maximal ideal M. Show that b_M's generate the unit ideal.